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# Stability-Augmentation System Gain Determination by Digital Computer

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**A digital program has been developed to compute the stability-augmentation system gain values yielding the best least squares fit of actual transfer function poles and zeros to desired pole-zero locations, for a set of vehicle transfer functions over a number of flight conditions. The desired locations, the transfer functions to be constrained, the flight conditions to be considered, the control loops to be used, and the gain programming constraints to be observed constitute the program input.**

## Nomenclature

| | |
|---|---|
| $A$ | = actual root value |
| $D$ | = desired root value |
| $KP$ | = roll-rate feedback to aileron |
| $KR$ | = yaw-rate feedback to rudder |
| $KCF$ | = stick crossfeed of roll command into rudder |
| $K_1$ | = constant |
| $L$ | = dummy index |
| $M$ | = number of steps to be used to obtain solution |
| $O$ | = initial value of actual root |
| $R$ | = sum of squares |
| $S$ | = computed step |
| $W$ | = computed weighing value |
| $X$ | = dependent variable |
| $\Delta(\ ), \delta(\ )$ | = finite increments in the quantity |

***Subscripts***

| | |
|---|---|
| $i$ | = flight condition designation |
| $j$ | = determinant designation (characteristic equation, etc.) |
| $k$ | = root component designation |
| $l$ | = dependent variable designation |
| $m$ | = specific numerical value for the dependent variables |

## Introduction

A TYPICAL aircraft stability - augmentation problem might be as follows:

1) As nearly as possible, place the complex zero pair of the roll-angle-to-aileron-deflection numerator and the roll subsidence, spiral divergence, and dutch roll roots of the lateral characteristic equation in specific $S$ plane locations.

2) Control these root locations over 28 flight conditions, as follows: Mach number: 0.2, 0.6, 0.8, 0.95, 1.2, 1.4, 1.76; angle of attack: 0°, 18° (at each Mach number); and dynamic pressure: 40, 120 psf (at each Mach number, $\alpha$ combination).

3) Obtain these root locations by scheduling the gains of the following three control loops. $KP$: roll-rate feedback to aileron; $KR$: yaw-rate feedback to rudder; and $KCF$: stick crossfeed of roll command into rudder.

4) Determine the root locations achievable if control gains are scheduled in any of three different ways, i.e.: a) Gains variable with only Mach number; b) gains variable with only angle of attack; and c) gains variable with both Mach number and angle of attack.

Therefore, with the numerator and characteristic equation in LaPlace form, the result is 56 polynomials (numerator and denominator at 28 flight conditions) whose coefficients are functions of numerical constants determined by the flight condition and three dependent variables ($KP$, $KR$, and $KCF$). The problem is to find those values of the dependent variables (for three methods of gain scheduling), which yield the best fit of the actual roots of this set of equations to the desired values. The next section derives the equations to be solved. The remainder of the paper discusses these equations, their mechanization on a digital computer, and the results obtained from applying the program to two example problems.

## Derivation of Equations

A typical equation root is symbolized by $(A_{ij1} + jA_{ij2})$ and its desired value by $(D_{ij1} + jD_{ij2})$. The square of the ab-

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solute value of the difference is thus

$$\begin{aligned}|(D_{ij1} + jD_{ij2}) - (A_{ij1} + jA_{ij2})|^2 &= |(D_{ij1} - A_{ij1}) + j(D_{ij2} - A_{ij2})|^2 \\ &= (D_{ij1} - A_{ij1})^2 + (D_{ij2} - A_{ij2})^2 \\ &= \sum_{k=1}^{2}(D_{ijk} - A_{ijk})^2 \end{aligned} \quad (1)$$

The sum of the squares of the absolute values of the differences for all flight conditions, determinants, and roots is then

$$\sum_i\sum_j\sum_k(D_{ijk} - A_{ijk})^2 \quad (2)$$

where the upper limit on the $(k)$ index is, in general, a function of $(j)$. To account for the situation where a small magnitude change in one root is as important as a larger change in another root, weighing factors must be associated with each root component difference. For the case where the elimination of all differences is equally important, the weighing factors would be computed as follows:

$$W_{ijk} = K_1/|D_{ijk} - O_{ijk}| \quad \ldots K_1 \text{ a const} \quad (3)$$

The expression for the sum of squares with weighing factors included is as follows:

$$R = \sum_i\sum_j\sum_k W_{ijk}^2(D_{ijk} - A_{ijk})^2 \quad (4)$$

Let the dependent variables appearing in the coefficients of the polynomials be symbolized by $(Xl)$ where $(l = 1, 2, 3, \ldots)$. No general expression for the individual roots of an arbitrary polynomial in terms of scalar variables $(Xl)$, appearing in its coefficients, exists. However, for specific numerical values of these variables (say the $m$th set of values), the magnitudes of the roots can be determined and will be symbolized as $(A_{ijkm})$. An approximate expression for the change in each root as a function of changes in the dependent variables about their $m$th values is given by

$$\Delta A_{ijkm} \approx \sum_{l=1}\left(\frac{\Delta A_{ijkm}}{\Delta Xl}\right)\delta(Xl) \quad (5)$$

This equation represents the approximate total derivative of $A_{ijkm}$ in terms of its approximate partial derivatives $(\Delta A_{ijkm}/\Delta Xl)$, and small parameter perturbations $\delta(Xl)$. Using this relation, an approximate expression for the roots in terms of the dependent variables can be written as follows:

$$A_{ijk} \approx A_{ijkm} + \sum_{l=1}\left(\frac{\Delta A_{ijkm}}{\Delta(Xl)}\right)\delta(Xl) \quad (6)$$

The accuracy of this expression improves with decreasing $\delta(Xl)$ values. The fact that it would be completely inadequate in the expression for $(R)$ is evident from Fig. 1. This figure shows that the first-order sensitivity of a typical root evaluated with augmentation gains zero can be completely unrelated to its value at a possible operational gain setting (desired value). The expression is adequate, however, if the approach illustrated in Fig. 2 is employed. Here the total difference between the desired and initial values of each root is broken into some arbitrary number of small steps (four, in the example). Instead of using the ultimately desired value of the roots in the expression for $(R)$, the first "desired" value would be used, and the $\delta(Xl)$ values that best approximate this change would be computed. The approximate expression for the roots with the new values of $(Xl)$ would be evaluated and the least squares equations solved using the second "desired" value of the root. The process would be continued until the ultimately desired value of the root was used for solution. Specifically, this process would be accomplished as follows.

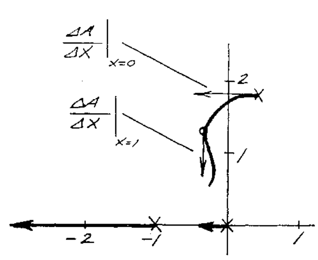



**Fig. 1 Example root loci.**

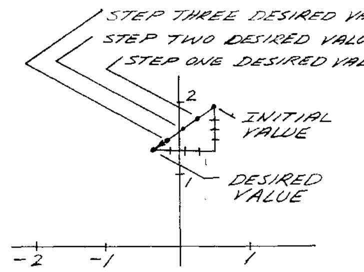



**Fig. 2 Illustration of step-wise approach.**

The increment in the desired value of each root is given by the following expression:

$$S_{ijk} = (D_{ijk} - O_{ijk})/M \quad (7)$$

Where $(M)$ is the number of steps to be used to obtain the final solution. The desired values of the roots for the $m$th step in the solution is given by

$$D_{ijkm} = O_{ijk} + m(S_{ijk}) \quad (8)$$

The values of the $(Xl)$ variables to be used to determine the constants in the expressions for $(A_{ijk})$ for the $m$th step are given by

$$(Xl)_m = (Xl)_0 + \sum_{m=1}^{m=m}\delta(Xl)_m \ldots \delta(Xl)_1 \quad \triangleq 0 \quad (9)$$

where $(Xl)_0$ symbolizes the initial value of the $l$th dependent variable (typically zero).

The sum of squares expression $(R)$ for the $m$th solution step $(R_m)$ can be obtained by substituting the appropriate expressions for the desired and actual values into the equation already obtained for $(R)$, i.e.,

$$R_m = \sum_i\sum_j\sum_k W_{ijk}^2\left[O_{ijk} + m(S_{ijk}) - A_{ijkm} - \sum_{l=1}\left(\frac{\Delta A_{ijkm}}{\Delta(Xl)}\right)\delta(Xl)\right]^2 \quad (10)$$

Since all quantities except $(m)$, $(\Delta Xl)$, and $(\delta Xl)$ are $(ijk)$ dependent, let these indexes be understood in the expressions that immediately follow, i.e.,

$$R_m = \sum_i\sum_j\sum_k W^2\left[O + m(S) - A_m - \sum_{l=1}\frac{\Delta A_m}{\Delta(Xl)}\delta(Xl)\right] \quad (11)$$

In these equations, $(O)$ symbolizes the initial value of an actual root and not the numerical value of zero. Minimizing $(R_m)$ with respect to the $\delta(Xl)$ variables implies:

$$\frac{\partial R_m}{\partial(\delta XL)} = 2\sum_i\sum_j\sum_k W^2\left[O + m(S) - A_m - \sum_{l=1}\frac{\Delta A_m}{\Delta(Xl)}\delta(Xl)\right]\left[-\frac{\Delta A_m}{\Delta(XL)}\right] = 0 \quad (12)$$

where $L = 1, 2, \ldots$, maximum value of $(l)$.

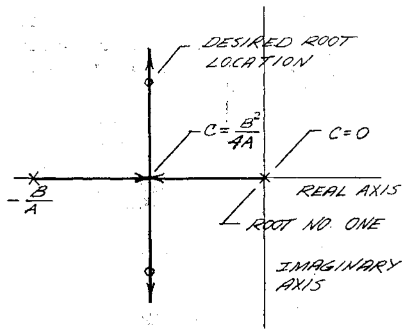



**Fig. 3a Root loci for quadratic with constant term as the root loci parameter.**

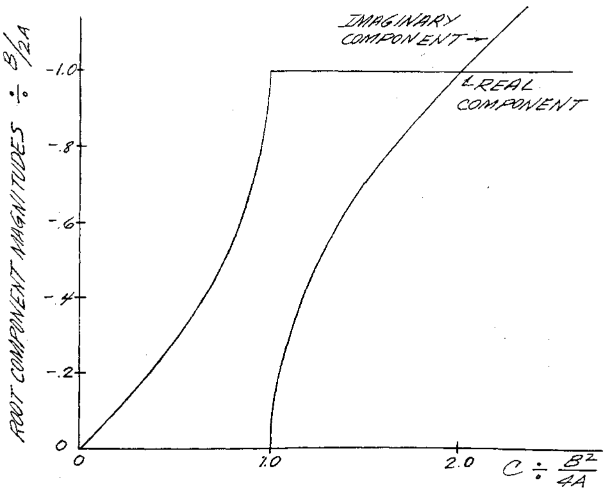



**Fig. 3b Variation of root number one components with root loci parameter values.**

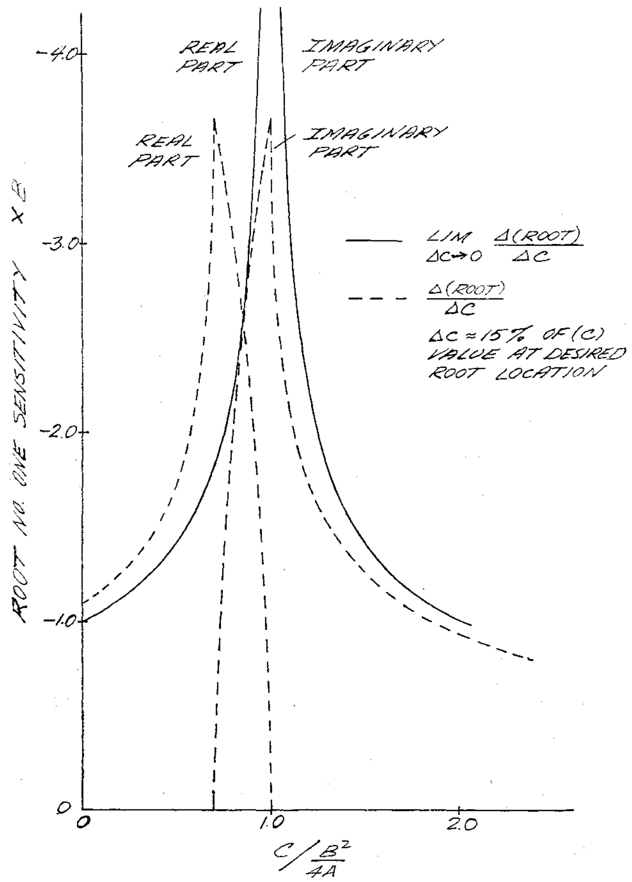



**Fig. 3c Root component sensitivities as a function of (*C*) and (Δ *C*) values.**

Alternately,

$$\sum_i \sum_j \sum_k W^2\left[O + m(S) - A_m\right]\left[\frac{\Delta A_m}{\Delta XL}\right] = \sum_i \sum_j \sum_k \sum_l W^2 \left[\frac{\Delta A_m}{\Delta Xl}\right]\left[\frac{\Delta A_m}{\Delta XL}\right]\delta(Xl) \quad (13)$$

where $L = 1, 2, \ldots, l_{\max}$. This last set of equations can be written in vector-matrix form as follows:

$$\mathbf{b}_L = (a_{Ll})\delta(\mathbf{Xl}) \quad (14)$$

where:

$$b_L = \sum_i \sum_j \sum_k W_{ijk}{}^2\left[O_{ijk} + m(S_{ijk}) - A_{ijkm}\right]\left[\frac{\Delta A_{ijkm}}{\Delta XL}\right] \quad (15)$$

$$a_{Ll} = \sum_i \sum_j \sum_k W_{ijk}{}^2\left[\frac{\Delta A_{ijkm}}{\Delta(XL)}\right]\left[\frac{\Delta A_{ijkm}}{\Delta(Xl)}\right] \quad (16)$$

The solution of this equation for the $\delta(\mathbf{Xl})$ vector components is given as $(a_{Ll})^{-1}\mathbf{b}_L$. Thus, the approach has been to divide the total changes desired in the roots into a number of steps and to use the equations derived to compute the gain changes required to accomplish each step; the sum of these changes representing the final solution.

## Simplified Numerical Example

For illustration purposes, the equations derived in the previous section will be applied to a greatly simplified numerical example. Given a single determinant at two flight conditions (i.e. two polynomials) as follows:

$$\text{at flight condition one} \ldots S^2 + 2S^1 + C = 0 \quad (17)$$

$$\text{at flight condition two} \ldots S^2 + 1S^1 + C = 0 \quad (18)$$

Let the desired root locations for both flight conditions be as follows:

$$S = -0.75 \pm j1.0 \quad (19)$$

In addition, let the initial values of the roots be those corresponding to $C = 0$, i.e.,

$$\text{for condition one:}\quad S_1 = 0. \qquad S_2 = -2 \quad (20)$$

$$\text{for condition two:}\quad S_1 = 0. \qquad S_2 = -1 \quad (21)$$

Assume that all root component differences are equally important and set $K_1 = 1$ (arbitrarily). Choosing the $(\Delta X1)$ value to be 0.2 and the number of solution steps ($M$) to be used as 3, the problem can be formulated in the previously adopted notation, as illustrated in Table 1.

For the first step of the solution, the appropriate root sensitivities are as follows: $\Delta A1111 = -0.105$, $\Delta A1121 = 0$, $\Delta A2111 = -0.2762$, $\Delta A2121 = 0$. The equation giving the gain change required to accomplish the first step is thus:

$$\delta(X1) = 0.8475/3.9 = 0.217 \quad (22)$$

where:

$$b1 = \frac{16}{9}\frac{(-1)}{4}\frac{(-0.105)}{0.2} + (1)\frac{(1)}{3}\frac{(0.)}{0.2} + \frac{16}{9}\frac{(-1)}{4}\frac{(-0.2762)}{0.2} + (1)\frac{(1)}{3}\frac{(0)}{0.2} = 0.8475 \quad (23)$$

**Table 1 Formulation of the problem**

| X1 = C | ΔX1 = 0.2 | $K_1$ = 1.0 | M = 3.0 |
|---|---|---|---|
| D111 = −0.75 | A1111 = 0111 = 0 | W111 = 1.33 | S111 = −0.25 |
| D112 = 1.00 | A1121 = 0112 = 0 | W112 = 1.00 | S112 = 0.33 |
| D211 = −0.75 | A2111 = 0211 = 0 | W211 = 1.33 | S211 = −0.25 |
| D212 = 1.00 | A2121 = 0212 = 0 | W212 = 1.00 | S212 = 0.33 |

$$a11 = \frac{(16)}{9}\left(\frac{-0.105}{0.2}\right)^2 + 0 + \frac{(16)}{9}\left(\frac{-0.2762}{0.2}\right)^2 + 0 = 3.9 \quad (24)$$

Using this gain value for the second step of the solution, the actual root values and their changes with $(\Delta X1)$ are as follows: $A1112 = -0.116$, $\Delta A1112 = -0.12$, $A1122 = 0.0$, $\Delta A1122 = 0.0$, $A2112 = -0.3184$, $\Delta A2112 = -0.1816$, $A2122 = 0.0$, and $\Delta A2122 = 0.409$. The second gain increment is then given by the following equation:

$$\delta(X1) = 2.073/6.31 = 0.328 \quad (25)$$

The gain value for the third step is thus 0.545 (= 0.217 + 0.328). Continuing in this fashion, the final result is $X1 = 1.72$, which can be easily verified to be the best mean square fit obtainable to the desired root values. For the particular choice of $(\Delta X1)$ magnitude used, two iterations on the final solution were required to obtain two decimal place accuracy (requiring a total of five solution steps).

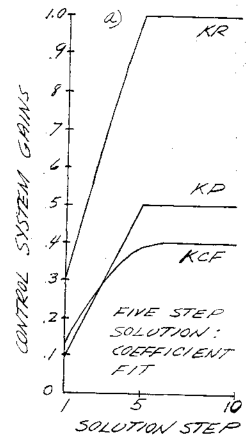
## Discussion of the Equations

Three factors bearing on either the validity or practicality of this approach are as follows: 1) $\Delta(Xl)$ magnitude selection, 2) $(M)$ magnitude requirements, and 3) matrix inversion error effects. The essential factors governing the selection of the magnitudes to be used for the $\Delta(Xl)$ values are evident from the simple example presented in Fig. 3. Figure 3a presents the root loci plot for a simple quadratic $(AS^2 + BS + C)$ with the constant term taken as the root locus parameter. From this plot, it is evident that the real and imaginary components of root number (1) will vary with $(C)$ as shown in Fig. 3b. The ratio of these root component changes to changes in $(C)$ for $(\Delta C)$ on the order of 10 to 20% of its final value, and for the limiting case when it approaches zero, are plotted in Fig. 3c. It can be seen from inspection of the least squares equations that, if $(\Delta C)$ values approaching zero were used in the region of a singularity of one of the root components, the magnitude of the corresponding derivative term would completely dominate the solution. Thus, the solution values of $\delta(Xl)$ would approach zero and the stepwise solution would hang up at this point. This problem is circumvented if values of $\Delta(Xl)$ on the order of 15% of the expected final value of the variables are used. The accuracy of the solution, using these relatively coarse intervals, has been found to be much better than is required in practice (three to four decimal places); this is not surprising when the relatively smooth variations of the root components are considered. As a matter of interest, in early check runs, the choice of $\Delta(Xl)$ magnitudes was varied by two orders of magnitude without appreciably affecting either solution times or accuracies.

The number of steps required for the root expression approximation to remain adequate does not appear to be a problem. In practical examples investigated to date, five steps appear to be sufficient and ten steps certainly would be. There are two reasons for this. First, airframe root loci are smooth functions. Second, there is an averaging-out effect in the solution computations since the sensitivities of perhaps 8–300 root components could be involved in a single solution. Thus, the errors in the individual sensitivities tend to sum toward zero in the result.

Matrix inversion errors can arise from both ill-conditioning and round-off errors. Ill-conditioning arises when one, or some linear combination of dependent variables, has very nearly the same effect upon the roots to be controlled as a second dependent variable (or second linear combination). The possibility of this occurring in practice decreases with increasing numbers of flight conditions to be considered and increasing numbers of roots to be constrained. If a singular, or near singular, matrix were to be encountered, it would be of more engineering usefulness to isolate and eliminate the redundant variable(s) than to attempt to achieve a more or less academic solution of the ill-conditioned system. The perturbations to the solution arising from round-off errors have been found to be completely negligible in the problems run to date. In summary, although there is no mathematical guarantee that numerical difficulties could not arise in the matrix inversion process, the physics of a practical airframe problem indicate that this is not too likely to happen.

**Fig. 4a Solution by coefficient fit program.**



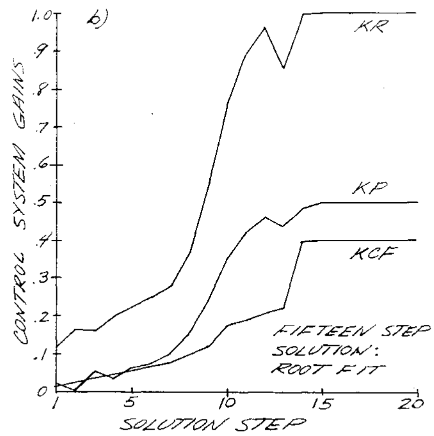




**Fig. 4b Solution by root fit program.**

## Alternate Mechanization

It is evident that the desired polynomial roots could be multiplied together to get desired values of the polynomial coefficients. Also, the coefficients can be expressed as a function of the dependent variables by an equation of the same form as that used to compute the polynomial roots. Thus, with relatively minor logic modifications, a variation on the root fit program was written which fitted to the polynomial coefficients (instead of the roots). This alternate mechanization has the advantage of faster solution times, principally because the root extraction process is eliminated. However, it has two disadvantages. First, it is no longer possible to weigh individual roots; the result is that small, but desirable, root changes have little influence on the solution obtained. Second, even though interest may center on only a few of the roots of a polynomial, it is necessary to place location requirements upon all the roots to establish the desired values of the polynomial coefficients. In summary, if all the roots were to be constrained and if an exact fit were possible with the dependent variables selected, both the coefficient and root fit programs would yield the same results, with the former being the most efficient.

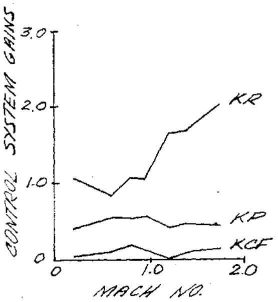



**Fig. 5 Gains variable with Mach number solution.**

## Program Input Format

The root fit program input consists of seven essential sections, as follows.

1) The linearized, transformed differential equations of the airframe in symbolic form.

2) The values of the scalars appearing in these equations for each flight condition to be considered.

3) Which transfer function numerators and denominators contain roots to be constrained.

4) The desired values of the roots to be constrained at each flight condition.

5) The scalars to be treated as dependent variables for the problem.

6) Gain programming instructions. For example, if the gains are to be variable only with Mach number in the first problem, the machine is instructed to group the flight condition data by Mach number and solve for the dependent variable values associated with each grouping.

7) The magnitudes of the changes in the dependent variables to be used to compute root sensitivities $\Delta(Xl)$, and the number of steps $(M)$ to be used in obtaining a solution to the problem.

## Example Applications

A checkout problem was run using the lateral-directional equations of a lifting body at six flight conditions: Constraints were placed upon the dutch roll, roll subsidence, and spiral divergence roots, in addition to the complex zeros appearing in the roll-angle-to-aileron-deflection numerator. The dependent variables were specified to be $(KP)$, $(KR)$, and

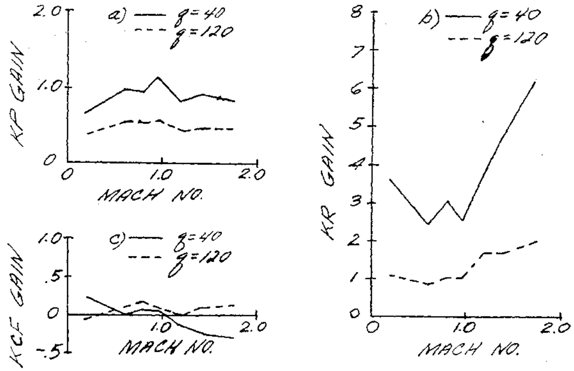



**Fig. 6 Solution for gains variable with Mach number and dynamic pressure.**

$(KCF)$. The desired values of the roots to be controlled were taken to be those values that would actually be obtained for $KP$, $KR$, and $KCF$ set, respectively, to 0.5, 1.0, and 0.4. The results for the best fit obtainable with constant gain settings are presented in Figs. 4a and 4b. In addition to the final solutions, these figures show the intermediate solutions obtained at each step in the step-wise solution process for both the coefficient and root fit programs.

In a more practical example, 28 flight conditions were specified for the same vehicle and dependent variables and the desired values of the roots were specified on the basis of yielding good handling qualities. The computed gain programs for gains variable with Mach number and with both Mach number and dynamic pressure are presented in Figs. 5 and 6, respectively. The computer time required by this particular example was on the order of 2 to 3 min.

## Summary and Conclusions

An approach to the direct calculation of stability augmentation system gain values, yielding the best fit of actual transfer function poles and zeros to the locations desired for them, has been developed in detail and has been demonstrated to be workable. The principal use of such a program is to reduce the time and effort required by a purely parametric approach to design.